Lecture 9:

4-10-18

Recall: We consider inhomogeneous eqt of the form

$$ay'' + by' + cy = r(t)$$

with $r(t) = e^{\alpha t}\cos\beta t\, P_k(t)$ or $e^{\alpha t}\sin\beta t\, P_k(t)$.

Step 1: Let $\gamma = \alpha + i\beta$, and

$$e^{\alpha t}\cos\beta t = \frac{1}{2}(e^{\gamma t} + e^{\bar{\gamma} t}) = \mathrm{Re}(e^{\gamma t})$$

$$e^{\alpha t}\sin\beta t = \frac{1}{2i}(e^{\gamma t} - e^{\bar{\gamma} t}) = \mathrm{Im}(e^{\gamma t})$$

We consider the equation with complex-valued inhomogeneous term $r(t) = e^{\gamma t}P_k(t)$, and look for $Y(t) = e^{\gamma t}Q_\ell(t)$ as a $\mathbb{C}$-valued particular solution

- i.e. $aY'' + bY' + cY = e^{\gamma t}P_k(t)$

$\mathbb{R}$-valued!

$$\rightsquigarrow \quad a\bar{Y}'' + b\bar{Y}' + c\bar{Y} = \overline{e^{\gamma t}P_k(t)}$$

$\Rightarrow$ Letting $\tilde{Y} = \frac{1}{2}(Y + \bar{Y})$ which is $\mathbb{R}$-valued

$$a\tilde{Y}'' + b\tilde{Y}' + c\tilde{Y} = e^{\alpha t}\cos\beta t\, P_k(t).$$

Similarly: $\hat{Y} := \frac{1}{2i}(Y - \bar{Y})$ solve

$$a\hat{Y}'' + b\hat{Y}' + c\hat{Y} = e^{\alpha t}\sin\beta t\, P_k(t)$$

Step 2: consider the inhomogeneous $\mathbb{C}$-valued term $e^{\gamma t}P_k(t)$, $P_k(t) = A_0 + A_1 t + \cdots + A_k t^k$ with $A_i \in \mathbb{C}$.

Trial: Let $Y(t) = e^{\gamma t}Q_\ell(t)$ with

$$Q_\ell(t) = B_0 + B_1 t + \cdots + B_\ell t^\ell.$$

with $B_i \in \mathbb{C}$ to be solved.

Equation:

$$\underbrace{a(j+2)(j+1)}_{\tilde{a}_{j+2}} B_{j+2} + \underbrace{(j+1)f'(\gamma)}_{\tilde{b}_{j+1}} B_{j+1} + \underbrace{f(\gamma)}_{\tilde{c}} B_j = A_j.$$

Case 1) if $f(\gamma) \neq 0$, then it is solvable with $\ell = k$

$$\begin{pmatrix} \tilde{c} & \tilde{b}_1 & \tilde{a}_2 & & 0 \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \tilde{a}_k \\ & & & \ddots & \tilde{b}_k \\ 0 & & & & \tilde{c} \end{pmatrix} \begin{pmatrix} B_0 \\ \vdots \\ B_k \end{pmatrix} = \begin{pmatrix} A_0 \\ \vdots \\ A_k \end{pmatrix}$$

Case 2) if $f(\gamma) = 0$ but $f'(\gamma) \neq 0$, then it is solvable with $\ell = k+1$.

$$\begin{pmatrix} \tilde{b}_1 & \hat{a}_2 & & \bigcirc \\ & \ddots & \ddots & \\ & & \ddots & \hat{a}_{k+1} \\ \bigcirc & & & \tilde{b}_{k+1} \end{pmatrix} \begin{pmatrix} B_1 \\ \vdots \\ B_{k+1} \end{pmatrix} = \begin{pmatrix} A_0 \\ \vdots \\ A_k \end{pmatrix}$$

Rk: There is no case for $f'(\gamma)=0$ since in the case of non-trivial complex root, $\gamma, \bar{\gamma}$ are distinct roots for $f(\gamma)=0$

($\because$ a, b, c are real!)

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§ Variation of parameter:

General equation:

$$y'' + p(t)y' + q(t)y = r(t) \qquad \mathbb{R}\text{-valued.}$$

suppose we know y_1, y_2 fundamental set of solution to

$$y'' + p(t)y' + q(t)y = 0$$

Q: How to find $r(t)$?

Idea: Trial: $Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$

be a guess.

Differentiate:

$$Y'(t) = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'$$

$$Y''(t) = u_1'' y_1 + 2u_1' y_1' + y_1'' + u_2'' y_1 + 2u_2' y_2' + u_2 y_2''$$

Plug into the eqt:

$$u_1(\cancel{y_1'' + p(t) y_1' + q(t) y}) + u_2(\cancel{y_2'' + p(t) y_1' + q(t) y_2})$$

$$+ u_1'' y_1 + 2u_1' y_1' + p(t) u_1' y_1 + u_2'' + 2u_2' y_2' + p(t) u_2' y_2$$

$$= r(t)$$

$$\Rightarrow u_1'' y_1 + u_2'' y_2 + 2u_1' y_1' + 2u_2' y_2' + p(t)(u_1' y_1 + u_2' y_2) = r(t)$$

Idea 1 Let $u_1' y_1 + u_2' y_2 = 0$

$$\Longrightarrow u_1'' y_1 + u_1' y_1' + u_2'' y_2 + u_2' y_2' = 0$$

$\therefore$ the equation simplify as.

$$u_1' y_1 + u_2' y_2 = 0$$

$$u_1' y_1' + u_2' y_2' = r(t)$$

i.e.

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ r \end{pmatrix}$$

with the matrix invertible!

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}^{-1} = W(t)^{-1} \begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix}$$

$$\therefore \quad \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = W(t)^{-1} \begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix} \begin{pmatrix} 0 \\ r \end{pmatrix}$$

$$= W(t)^{-1} \begin{pmatrix} -ry_2 \\ ry_1 \end{pmatrix}$$

redundant!

$$\Rightarrow \quad \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \int W(t)^{-1} \begin{pmatrix} -ry_2 \\ ry_1 \end{pmatrix} dt + \cancel{\begin{pmatrix} d_1 \\ d_2 \end{pmatrix}}$$

$\therefore$ General solution:

$$y(t) = c_1 y_1 + c_2 y_2 - y_1 \int \frac{ry_2}{W}\, dt + y_2 \int \frac{ry_1}{W}\, dt$$

Example:

$$y'' - 3y' + 2y = \frac{e^{3t}}{e^t + 1}.$$

Step 1: Solve the homogeneous equation

$$y'' - 3y' + 2y = 0$$

Char eqt: $\quad r^2 - 3r + 2 = 0$

$$\Rightarrow \quad r_1 = 2, \quad r_2 = 1$$

$$y_1 = e^{2t}, \quad y_2 = e^{t}.$$

$$W(t) = \det\begin{pmatrix} e^{2t} & e^{t} \\ 2e^{2t} & e^{t} \end{pmatrix} = -e^{3t}$$

Step 2:

Compute $\displaystyle\int \frac{r y_1}{W}\,dt$

$$= \int \frac{e^{3t}}{e^{t}+1}\cdot\frac{e^{2t}}{-e^{3t}}\,dt$$

$$= \int \frac{-e^{2t}}{e^{t}+1}\,dt = -\int \frac{(e^{t}-1)(e^{t}+1)}{e^{t}+1}\,dt - \int \frac{1}{e^{t}+1}\,dt$$

$$= t - e^{t} - \log(e^{t}+1) + \cancel{c_1}^{\nearrow 0}.$$

$$\int \frac{r y_2}{W}\,dt = \int \frac{-e^{t}}{e^{t}+1}\,dt = -t + \log(e^{t}+1)$$

$$\therefore\; Y(t) = e^{2t}\left(-t + \log(e^{t}+1)\right) + e^{t}\left(t - e^{t} - \log(e^{t}+1)\right)$$